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Forms of Lepage type and the balance systems

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A B S T R A C T

In this work we apply infinitesimal variational calculus to the systems of balance equations. We determine a class of the exterior $n+(n+1)$-forms Θ on the jet bundle of infinite order over a configurational bundle $\pi : Y^{n+m} \to X^n$ similar to the class of Lepage n-forms. Systems of differential equations obtained in the way similar to one used in the Lagrangian field theory, include the Euler–Lagrange equations corresponding to a Lagrangian functions as well as arbitrary regular systems of balance equations. For a balance system with a symmetry group G we present the Noether balance laws corresponding to the generators of the Lie algebra of the group G.

1. Introduction

Dynamical equations for the physical fields of continuum thermodynamical systems are almost always formulated as the systems of balance equations (balance systems). Such a system for the fields $y^i(x^\mu)$, $i=1,\dots,m$; $\mu=0,1,2,3$, has the form

$$F^{\mu}_{i,\mu} = \Pi_i, \quad i=1,\dots,m. \tag{1.1}$$

Quantities: F^0_i — density in i-th equation, F^A_i — flux components in i-th equation and Π_i — source+production term in i-th equations are functions of space–time coordinates x^μ, values of fields $y^i(x)$ and their derivatives $z^i_\Lambda(x)$ (see notations below) up to some order: $|\Lambda| \leqslant k$.

Only rarely these balance equations can be presented as the Euler–Lagrange equations for some conventional variational problem with a Lagrangian L. In the works [13,14] it was showed that any balance system can be obtained as the system of Euler–Lagrange equations of an *infinitesimal* variational problem determined by the *proper version of Poincaré–Cartan form* Θ (including the source/production term, see Section 3 below or [13,14]). Yet, the price one has to pay for this is that, in difference to the conventional variational problems defined by an action $\int_D L\,d^4x$, variational vector fields ξ participating in the separating Euler–Lagrange equations are subject to the restrictions [14]. These restrictions prevent the use of the technique of infinitesimal variational calculus in its full strength. That is why, in the paper [15] we suggested to modify the Poincaré–Cartan form on the 1-jet bundle $J^1(\pi)$ by a 1-contact term removing the restriction to the use of arbitrary variations (see Section 3 below).

On the other hand, the same system of Euler–Lagrange equations of the classical field theory can be obtained by using the other dynamical (Lepage) forms on the jet bundles $J^k(\pi)$ of which the Poincaré–Cartan form is one (however important) choice [7].

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Thus, it seems natural to determine, if the modification of Poincaré–Cartan form suggested in [15] is a chance fact or an example of a more general construction leading to a natural class of dynamical equations containing conventional Euler–Lagrange equations as well as the regular (i.e. such that the number of balance equations is equal to the number of dynamical fields) systems of balance equations of the classical field theory. Positive answer to this question is given in this work. We determine the class of dynamical $(n+(n+1))$-forms Θ that produce a systems of PDE by the standard procedure of infinitesimal variational calculus:

$$J^k(s)^* i_\xi\, d\Theta = 0,$$

admitting arbitrary variational vector fields ξ.

The condition characterizing these exterior forms is similar to the one defining the Lepage forms (see [8]), so, it is natural to call these forms the "forms of Lepage type".

In Section 2 we introduce the basic notions and notations used in the paper. In Section 3 the results of papers [13,14] leading to the definition of the forms of Lepage type are reviewed. In Section 4 we introduce the exterior forms of Lepage type on the jet bundle of infinite order over a configurational bundle $\pi : Y \to X$, get the balance system corresponding to these forms and present two examples of these forms corresponding to the important classes of balance systems of continuum thermodynamics. In Section 5 a subclass of forms of Lepage type is defined and it is shown that any balance system can be realized by a form of this type (called ML-forms). It is shown that any form of Lepage type of order 1 can be decomposed into the sum of Poincaré–Cartan form corresponding to a Lagrangian L and the ML-form of Lepage type. In Section 6 we prove the first Noether theorem for the forms of Lepage type and their infinitesimal symmetries. In Appendix A we present definition and basic properties of the operator of con-differential used for invariant formulation of balance systems.

2. Settings and notations

Throughout this paper $\pi : Y \to X$ will be a (*configurational*) fibred bundle with an n-dim connected paracompact smooth (C^∞) manifold X as the base and a total space Y, $\dim(Y) = n+m$. The fiber of the bundle π is an m-dim connected smooth manifold U.

The base manifold X is endowed with a pseudo-Riemannian metric G. The volume form of metric G will be denoted by η. In this paper we will not be dealing with the boundary of a base manifold X, in applications X can be considered as an open subset of R^n or as a compact manifold.

We will be using fibred charts (W, x^μ, y^i) in a domain $W \subset Y$. Here $(\pi(W), x^\mu)$ is a chart in X and y^i are coordinates along the fibers. Local frame corresponding to the chart (W, x^μ, y^i) will be denoted by $(\partial_\mu = \partial_{x^\mu}, \partial_i = \partial_{y^i})$ (shorter notation will be used in more cumbersome calculations) and the corresponding coframe – (dx^μ, dy^i).

Introduce the contracted forms $\eta_\mu = i_{\partial_{x^\mu}} \eta$, $\eta_{\mu\nu} = i_{\partial_{x^\nu}} i_{\partial_{x^\mu}} \eta$. Below we will be using following relations for the forms $\eta_\nu, \eta_{\nu\mu}$ (here and below $\lambda_G = \ln(\sqrt{|G|})$):

$$\begin{cases} dx^\nu \wedge \eta_\mu = \delta^\nu_\mu \eta, \\ dx^\nu \wedge \eta_{\mu\sigma} = \delta^\nu_k \eta_\mu - \delta^\nu_\mu \eta_\sigma, \\ d\eta_\mu = \lambda_{G,\mu} \eta, \\ d\eta_{\mu\nu} = (\lambda_{G,\nu} \eta_\mu - \lambda_{G,\mu} \eta_\nu). \end{cases} \tag{2.1}$$

Sections $s : V \to Y$, $V \subset X$ of the bundle π represent the collection of (classical) fields y^μ defined in the domain $V \subset X$. Usually these fields are components of some tensor fields or tensor densities fields (sections of "natural bundles" [2]).

For a manifold M we will denote by $\pi_* : T(M) \to M$ the tangent bundle of a manifold M, by $V(M) \subset T(M)$ the subbundle of vertical vectors, i.e. vectors $\xi \in T_y(M)$ such that $\tau_{M*y}\xi = 0$. The sheaf of smooth vector fields on M (sections of the tangent bundle) will be denoted by $\mathcal{X}(M)$.

Denote by $\Omega^r \to M$ the bundle of exterior r-forms on the manifold M and by $(\Omega^* = \bigoplus_{r=0}^\infty \Omega^r(M))$ the differential algebra of exterior forms on the manifold M.

Given a fiber bundle $\pi : Y \to X$ denote by $J^k(\pi)$ the k-jet bundle of sections of the bundle π [2,5,17]. Denote by $\pi_{kr} : J^k(\pi) \to J^r(\pi)$, $k \geqslant r \geqslant 0$, the natural projections between the jet bundles of different order and by $\pi_k : J^k(\pi) \to X$ the projection to the base manifold X. Projection mappings $\pi_{k(k-1)} : J^k(\pi) \to J^{k-1}(\pi)$ in the tower of k-jet bundles

$$\cdots \to J^k(\pi) \to J^{k-1}(\pi) \to \cdots \to Y \to X$$

are *affine bundles* modeled by the vector bundle $\bigwedge^k T^*(X) \otimes_{\to J^{k-1}(\pi)} V(\pi) \to J^{k-1}(\pi)$.

Denote by $J^\infty(\pi)$ the infinite jet bundle of bundle π – inverse limit of the projective sequence $\pi_{k(k-1)} : J^k(\pi) \to J^{k-1}(\pi)$. Space $J^\infty(\pi)$ is endowed with the structure of inverse limit of differentiable manifolds with the natural sheaves of vector fields, differential forms, etc. making the projections $\pi_{\infty k} : J^\infty(\pi) \to J^k(\pi)$ smooth surjections. See [5,17] for more about structure and properties of k-jet bundles.

For a multi-index $I=\{i_1,\dots,i_n\}$, $i_k\in\mathbb{N}$, denote by ∂^I the differential operator in $C^\infty(X)$ $\partial^I f=\partial^{i_1}_{x^1}\cdot\ldots\cdot\partial^{i_n}_{x^n}$. To every fibred chart (W,x^μ,y^i) in Y there corresponds the fibred chart $(x^\mu,y^i,z^i_\mu,|I|=\sum_s i_s\leqslant k)$ in the domain $W^k=\pi^{-1}_{k0}(W)\subset J^k(\pi)$. Coordinates z^i_Λ in the fibers of jet bundles are defined by the condition $z^\mu_I(j^k_x s)=\partial^I s^\mu(x)$.

For $k=1,\dots,\infty$ the space $J^k(\pi)\to X$ of k-jet bundle is endowed with the **contact (Cartan) distribution** Ca^k defined by the basic **contact forms**

$$\omega^i=dy^i-z^i_\mu\,dx^\mu,\dots,\omega^i_I=dz^i_I-z^i_{I+1_\nu}\,dx^\nu,\quad |I|<k. \tag{2.2}$$

These forms generate the contact ideal $C^k\subset\Omega^*(J^k(\pi))$ in the algebra of all exterior forms. Denote by $I(C^k)$ the differential ideal of contact forms. This ideal is generated by the basic contact forms (2) and by the forms dz^i_I, $|I|=k$, see [8]. In the case where $k=\infty$ the basic contact forms dz^i_I are absent form the list of generators of ideal $I(C^k)$.

A p-form is called l-contact if it belongs to the l-th degree of this ideal $(C^k)^l\subset\Omega^*(J^k(\pi))$. 0-contact exterior forms are also called **horizontal (or π_k-horizontal) forms** (or, sometimes, *semi-basic forms*). We denote by $k\,$Con the k-contact forms that appear in calculations. For $k=1$ we will omit index 1.

Let $0\leqslant s<k$. A form $\nu\in\Omega^*(J^k(\pi))$ is called π_{ks}-**horizontal** if it belongs to the subalgebra $C^\infty(J^k(\pi))\pi^*_{ks}\Omega^*(J^s(\pi))\subset\Lambda^*(J^k(\pi))$.

Remind now the following basic result (D. Krupka, [6])

Theorem 1. *Let a form $\nu\in\Omega^q(J^k(\pi))$ be $\pi_{k(k-1)}$-horizontal (i.e. $\nu=\pi_{k(k-1)}\nu_*$, $\nu_*\in\Omega^q(J^{k-1}(\pi))$). Then there is unique* contact *decomposition of the form ν*

$$\nu=\nu_0+\nu_1+\dots+\nu_q, \tag{2.3}$$

where ν_i, $0\leqslant i\leqslant q$, is an i-contact form on $J^k(\pi)$. Form ν_0 is called the horizontal part of the form ν_* (and of the form ν as well).

Related to the contact decomposition is the decomposition of differential operator d as the sum of **horizontal and vertical differentials** d_h, d_V: for a q-form $\nu\in\Omega^*(J^k(\pi))$ its differential $d\nu$ lifted into the $J^{k+1}(\pi)$ is presented as the sum of horizontal and contact (vertical) terms: $d\nu=d_h\nu+d_V\nu$, see [6,8]. Operators d_h, d_V are naturally defined in the space $\Omega^*(J^\infty(\pi))$.

We recall that these operators have the following homology properties

$$d_h^2=d_V^2=d_Vd_h+d_hd_V=0.$$

In particular, for a function $f\in C^\infty(J^\infty(\pi))$ (depending on the jet variables z^i_I up to some degree, say $|I|\leqslant k$),

$$df=(d_\mu f)\,dx^\mu+\sum_{|I|\geqslant 0} f_{,z^i_I}\omega^i_I, \tag{2.4}$$

where

$$d_\mu f=\partial_{x^\mu}f+\sum_{I||I|\geqslant 0} z^i_{I+1_\mu}\partial_{z^i_I}f \tag{2.5}$$

is the **total derivative** of the function f by x^μ. The series in the formulas (2.4)–(2.5) contains finite number of terms: $|I|\leqslant k$.

3. Lifted Poincaré–Cartan form of a balance system and its contact source modification

In the works [13,14] it was shown that any balance system (for convenience, we will denote this system be ★) of order k for m dynamical fields y^i

$$d_\mu F^\mu_i\big(j^ks\big)+F^\mu_i\big(j^ks\big)\lambda_{G,\mu}=\Pi_i\big(j^ks\big),\quad i=1,\dots,m,\qquad (\bigstar) \tag{3.1}$$

can be realized in invariant form — as the requirement that for many enough variational vector fields $\xi\in\mathcal{X}(J^k(\pi))$,

$$j^{k+1}s^*i_\xi\tilde{d}\Theta=0. \tag{3.2}$$

Here $\tilde{d}$ is the "con-differential", see Appendix A. The form

$$\Theta=F^\mu_i\omega^i\wedge\eta_\mu+\Pi_i\omega^i\wedge\eta \tag{3.3}$$

is the $n+(n+1)$-form on the k-jet bundle $J^k(\pi)$ — analog of the Poincaré–Cartan form for the balance system (3.1) (these forms were called *lifted* in [13–15]). Yet, calculation of $i_\xi\tilde{d}\Theta$ shows that unless the vector field ξ satisfies certain conditions, obtained expression will depend on the derivatives of the vector field ξ and, as a result, cannot be used for separation of components of the balance system. It was shown that locally there are always enough variational vector fields $\xi \in \mathcal{X}(Y)$ satisfying this condition and, therefore, ensuring the separation of equations. Yet, by reasons provided in the Introduction it is highly inconvenient to have such restriction to the vector fields used in combination with the Poincaré–Cartan form (3.3).

That is why, in the work [15] we suggested to modify the Poincaré–Cartan form (3.3) on the jet bundle $J^k(\pi)$ for $k > 1$ and on the 2-jet bundle $J^2(\pi)$ for $k = 1$ by adding an extra source term:

$$\widetilde{\Theta} = F^{\mu}_i \omega^i \wedge \eta_\mu - \Pi_i \omega^i \wedge \eta - F^{\mu}_i \omega^i_\mu \wedge \eta. \tag{3.4}$$

We call such a form — the ML-form (modified, lifted, see [15]) Poincaré–Cartan form.

Consider the lift ξ^2 to the bundle $J^2(\pi)$ of a vector field $\xi \in \mathcal{X}(J^1(\pi))$ for $k = 1$ and the lift $\xi^{k+1} \in \mathcal{X}(J^{k+1}(\pi))$ of the vector field ξ^k on $J^k(\pi)$ for $k > 1$. Applying to the form $\widetilde{\Theta}$ the arguments leading to the representation (3.2) of the balance system, we get

$$i_{\xi^{k+1}} \tilde{d}\widetilde{\Theta} = -\omega^i(\xi)\big[d_\mu F^{\mu}_i + \lambda_{G,\mu} F^{\mu}_i - \Pi_i\big]\eta + \mathrm{Con}. \tag{3.5}$$

This proves the following

Theorem 2. *For a given balance system (3.1) of order $k \geqslant 0$ and a section $s : W \to Y$ of the bundle π the following statements are equivalent*:

(1) *For any vector field $\xi \in \mathcal{X}(J^k(\pi))$ and its arbitrary prolongation to the π^{k+1}_k-projectable vector field ξ^{k+1} on $J^{k+1}(\pi)$,*

$$j^{k+1\,*}(s) i_{\xi^{k+1}} \tilde{d}\widetilde{\Theta} = 0. \tag{3.6}$$

(2) *Balance equations* ★: $(F^{\mu}_i \circ j^k(s))_{;\mu} = \Pi_i \circ j^{k+1}(s)$, $i = 1, \ldots, m$, *holds.*

Corollary 1. *Any balance system (3.1) of order $k \geqslant 0$ can be presented in the infinitesimal variational form (3.2) using lifted Poincaré–Cartan form (3.3) and admissible variations of dynamical fields y^i or using ML-form (3.4) and arbitrary variations of dynamical fields y^i.*

4. Forms of the Lepage type and the balance systems

In this section we introduce the class of exterior $(n + (n + 1))$-forms on the jet bundles $J^k(\pi)$ following the definition of Lepage form in the Lagrangian formalism [7,8]. These forms include the Poincaré–Cartan ML-forms defined in the previous section and those defined by a Lagrangian L. Essential property of these forms in the possibility to modify the production term $\Pi_i \omega^i \wedge \eta$ in order to allow arbitrary variations of the fields y^i and their prolongations. Possibility to use arbitrary variations in obtaining the system of balance equations is highly valuable for studying symmetries and related conservation and balance laws (see below, Section 6). We consider all the forms, functions and vector fields defined on $J^\infty(\pi)$ assuming, conventionally, that the all exterior forms (and functions) depend on the values of fields and their derivatives up to some finite order, i.e are defined on $J^k(\pi)$ for some finite k.

Consider an arbitrary $n + (n + 1)$-form in $J^\infty(\pi)$. Using the contact decomposition (2.3), one can present this form as follows

$$\Theta = \check{L}\eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta_\mu - \sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi^{\Lambda}_i \omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con}. \tag{4.1}$$

We assume that this form belongs to the subspace $\Omega^*(J^k(\pi))$ for some $k \geqslant 0$. Minimal k such that $\Theta \in \Omega^*(J^k(\pi))$ is called the order of the form Θ (and of the corresponding system of PDE).

We are using the notation $\check{L}$ in the representation (4.1) to demonstrate the parallelism with the contact representation of a Lepage form of a Lagrangian form $L\eta$, [8].

Calculate the con-differential $\tilde{d}\Theta$ (see Appendix A):

$$\begin{aligned}\tilde{d}\Theta = {}& d\check{L} \wedge \eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} dF^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta_\mu + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i d\omega^i_\Lambda \wedge \eta_\mu - \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_\Lambda \wedge d\eta_\mu \\ & + \sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi^{\Lambda}_i \omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con}.\end{aligned} \tag{4.2}$$

We have $d\check{L} \wedge \eta = d_h\check{L} \wedge \eta + d_v\check{L} \wedge \eta = d_v\check{L} \wedge \eta = \sum_{i,\Lambda,|\Lambda|\geqslant 0} \check{L}_{,z^i_\Lambda} \omega^i_\Lambda \wedge \eta$, and

$$\begin{aligned} dF^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta_\mu &= (d_h + d_v) F^{\Lambda\mu}_i \wedge \omega^i_\Lambda \wedge \eta_\mu \\ &= \big(d_\sigma F^{\Lambda\mu}_i\big)\, dx^\sigma \wedge \omega^i_\Lambda \wedge \eta_\mu + 2\,\mathrm{Con} \\ &= -d_\mu\big(F^{\Lambda\mu}_i\big)\omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con}. \end{aligned} \tag{4.3}$$

Using (2.3) and the formula

$$d\omega^i_\Lambda \wedge \eta_\mu = -\omega^i_{\Lambda\sigma}\, dx^\sigma \wedge \eta_\mu = -\omega^i_{\Lambda\mu} \wedge \eta$$

(valid in $J^\infty(\pi)$) and extracting terms with $|\Lambda| = 0$ we continue:

$$\begin{aligned}\tilde{d}\Theta &= \sum_{\Lambda,|\Lambda|\geqslant 0} \check{L}_{,z^i_\Lambda}\omega^i_\Lambda \wedge \eta - \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} d_\mu F^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta - \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} \lambda_{G,\mu} F^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta \\ &\quad - \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_{\Lambda\mu} \wedge \eta + \sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi^\Lambda_i \omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con} \\ &= \left[\check{L}_{,y^i} - d_\mu F^\mu_i - \lambda_{G,\mu} F^\mu_i + \Pi_i\right]\omega^i \wedge \eta + \sum_{i,\mu,\Lambda,|\Lambda|>0} \left[\check{L}_{,z^i_\Lambda} - d_\mu F^{\Lambda\mu}_i - \lambda_{G,\mu} F^{\Lambda\mu}_i + \Pi^\Lambda_i\right]\omega^i_\Lambda \wedge \eta \\ &\quad - \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_{\Lambda\mu} \wedge \eta + 2\,\mathrm{Con}. \end{aligned} \tag{4.4}$$

In the last term we introduce multi-index $\Xi = \Lambda\mu$. Every index Ξ such that $|\Xi| = |\Lambda| + 1 = k + 1$ can be written as $\Lambda\mu$ in a variety of ways, choosing μ to be one of indices entering Ξ. Presentation $\Xi = \Lambda\mu$ can be achieved by $n_{|\Xi|}$ times. Number $n_{|\Xi|}$ is equal to the number of nonzero terms λ_i in the representation of multi-index $\Xi = (\lambda_1, \dots, \lambda_n)$. Thus,

$$\sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_{\Lambda\mu} \wedge \eta = n_{|\Xi|} \sum_{i,\Xi,|\Xi|>0} F^\Xi_i \omega^i_\Xi \wedge \eta.$$

Changing notation back to Λ we finally get the expression for the con-differential of the form Θ:

$$\begin{aligned}\tilde{d}\Theta &= \left[\check{L}_{,y^i} - d_\mu F^\mu_i - \lambda_{G,\mu} F^\mu_i + \Pi_i\right]\omega^i \wedge \eta \\ &\quad + \sum_{i,\Lambda,|\Lambda|>0} \left[\check{L}_{,z^i_\Lambda} - \sum_\mu \left(d_\mu F^{\Lambda\mu}_i + \lambda_{G,\mu} F^{\Lambda\mu}_i\right) + \Pi^\Lambda_i - n_{|\Lambda|} F^\Lambda_i\right]\omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con}. \end{aligned} \tag{4.5}$$

Definition 1. A form (4.1) is called the "**form of Lepage type**" if the following equivalent statements are true:

(1) The 1-contact term of the lift $\pi^*_{(k+1)k}\tilde{d}\Theta$ is $\pi_{(k+1)0}$-horizontal.
(2) For any $\pi_{k+1,0}$-vertical vector field ξ, $h(i_\xi \tilde{d}\Theta) = 0$.
(3) For all (i, Λ) such that $|\Lambda| > 0$,

$$\check{L}_{,z^i_\Lambda} - \sum_\mu \left(d_\mu F^{\Lambda\mu}_i + \lambda_{G,\mu} F^{\Lambda\mu}_i\right) + \Pi^\Lambda_i - n_{|\Lambda|} F^\Lambda_i = 0. \tag{4.6}$$

Proof. The proof of the equivalence of conditions 1, 2, 3 is similar to the proof of corresponding statement in the Lagrangian Variational Calculus, see [8].

If condition (3) in the definition holds, then, $\tilde{d}\Theta = [\check{L}_{,y^i} - d_\mu F^{\Lambda\mu}_i - \lambda_{G,\mu} F^{\Lambda\mu}_i + \Pi_i]\omega^i \wedge \eta$ and the conditions (1), (2) hold as well since for any $\pi_{k+1,0}$-vertical vector field ξ, $i_\xi(\omega^i \wedge \eta) = 0$.

Conditions (1) and (2) are equivalent — first condition says that 1-contact part of $\tilde{d}\Theta$ may contain only ω^i of all basic contact 1-forms and the second states the same in other language.

To prove that (3) follows from (1) or (2) we notice that the contact 1-forms ω^i_Λ are linearly independent and that the vector fields $\partial_{z^i_\Lambda}$ separate them. Thus, let the coefficient of the 1-contact form $\omega^j_\Pi \wedge \eta$ is not equal zero at some point. Then the expression $i_{\partial_{z^j_\Psi}} \tilde{d}\Theta = [\check{L}_{,z^j_\Psi} - \sum_\mu (d_\mu F^{\Psi\mu}_j + \lambda_{G,\mu} F^{\Psi\mu}_j) + \Pi^\Psi_j - n_{|\Pi|} F^\Lambda_j]\eta + \mathrm{Con}$ has nonzero horizontal part at this point. This contradicts to the condition (2).

Since the statements (2) and (3) are equivalent, the proof is finished. □

Theorem 3. *Let*

$$\Theta = \check{L}\eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F^{\Lambda\mu}_i \omega^i_\Lambda \wedge \eta_\mu - \sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi^\Lambda_i \omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con} \tag{4.7}$$

be an $n + (n+1)$-form of Lepage type. Then the following statements are equivalent for a section $s : X \to Y$

(1) *For all vector fields $\xi \in \mathcal{X}(J^\infty(\pi))$,*

$$j^k s^* i_\xi \tilde{d}\Theta = 0. \tag{4.8}$$

(2) *Section s is the solution of the following system of equations*

$$d_\mu F_i^\mu + \lambda_{G,\mu} F_i^\mu = \check{L}_{,y^i} + \Pi_i, \quad i = 1, \dots, m. \tag{4.9}$$

Remark 1. Expression in the right side of (4.10) looks like a splitting of the source/production part into the potential term (similar to the term $L_{,y^i}$ in the conventional Euler–Lagrange equations) and the complemental term.

4.1. Examples

Here we provide two examples of the forms of Lepage type corresponding to the types of balance systems used in the continuum thermodynamics, illustrating the scope of introduced notion.

(1) **Balance systems of (L, Ψ) type.** In the works on continuum thermodynamics there were suggested different models describing evolution of the physical systems with dissipation (see, for instance, [9,10]). One of the most popular types of such models is one that is constructed using a Lagrangian L and the dissipative potential Ψ. Here $n = 4$, $x^0 = t$ is the time, x^A, $A = 1, 2, 3$, are physical or material coordinates in the material. Both functions L, Ψ depend on the fields y^i and their derivatives up to an order k but it is dependence of Ψ on the rates z_0^i of evolution of the fields y^i plays the decisive role — the system of equations corresponding to the couple (L, Ψ) is

$$\frac{\delta L}{\delta y^i} = \frac{\partial \Psi}{\partial z_0^i}, \quad i = 1, \dots, m. \tag{4.10}$$

This balance system admits the invariant formulation (4.9) with the form of Lepage type

$$\Theta_{L,\Psi} = L\eta + L_{,z_\mu^i}\omega^i \wedge \eta_\mu - \Psi_{,z_0^i}\omega^i \wedge \eta.$$

(2) **Balance systems of Godunov type.**
These balance systems are defined by n functions $L^\mu(y^i)$, in applications $n = 4$, $\mu = 0, 1, 2, 3$. Balance system has the form

$$d_\mu\left(\frac{\partial L^\mu}{\partial y^i}\right)(j^1 s) = \frac{\partial^2 L^\mu}{\partial y^i \partial y^j} y^j_{,x^\mu}(j^1 s) = 0, \quad i = 1, \dots, m. \tag{4.11}$$

If the function L^0 is convex, this system is the symmetrical hyperbolic system of Friedrichs type [3].
Balance systems of this type were introduced by S. Godunov in [4]. Later on they were studied by G. Boillat [1] and by T. Ruggeri and his collaborators (see [16,11]). At the work [16] it was proved that any balance system of order 0 (so that F_i^μ, Π_i depend on the fields y^i but not on their derivatives) admitting the entropy balance law with the convex entropy density can be transformed to the system of the form (4.12).
The form of Lepage type, corresponding to the system (4.12) is

$$\Theta_G = L^\mu_{,y^i}\omega^i \wedge \eta_\mu = d_\nu L^\mu \wedge \eta_\mu.$$

5. Special cases

Example 1. Consider the condition (4.7) for the form Θ of order 1, i.e. such that $F_i^\Lambda, \Pi_i^\Lambda = 0$ for all Λ such that $|\Lambda| > 1$. Thus, the form Θ is defined on the k-jet bundle $J^k(\pi)$, $l \geqslant 2$, due to the presence of ω^i_μ. Then, the representation (4.1) of an $n + (n + 1)$-form Θ reduces to

$$\Theta = \check{L}\eta + \sum_{i,\mu} F_i^\mu \omega^i \wedge \eta_\mu - \sum_i \Pi_i \omega^i \wedge \eta - \sum_{i,\mu} \Pi_i^\mu \omega^i_\mu \wedge \eta + 2\,\mathrm{Con}. \tag{5.1}$$

Condition (4.7) of Definition 1 takes the form

$$\begin{cases} \Pi_i^\lambda = F_i^\lambda - \check{L}_{,z^i_\lambda}, & \forall(\lambda, i), \\ \check{L}_{,z^i_\Lambda} = 0, & \forall(\Lambda, i),\ |\Lambda| > 1. \end{cases} \tag{5.2}$$

Thus, due to the second of conditions (5.2), function $\check{L}$ does not depend on z^i_Λ for $|\Lambda| > 1$. Functions Π_i^λ do not enter the balance system (4.10) and can be chosen arbitrarily.

Conditions (5.2) are obviously satisfied in the following two cases:

(1) Lagrangian case: $\Pi_i^\lambda = 0$, $F_i^\lambda = \check{L}_{,z^i_\lambda}$.
(2) Case of modified lifted Poincaré–Cartan form (ML-form) $\tilde{\Theta}$: $\check{L} = 0$, $\Pi_i^\lambda = F_i^\lambda$.

It follows from this that

Proposition 1. *For any balance system* (4.10) *of order* 1,

(1) *There exists unique,* mod 2 Con *forms, form Θ of Lepage type of order* 1 *with $\check{L} = 0$ producing this system. This is the ML Poincaré–Cartan form* (3.4)*.*
(2) *For any representation of the production terms Π_i as the sum of* potential *and complemental parts $\Pi_i = K_{,y^i} + Q_i$ there exist unique,* mod 2 Con *forms, Lepage type form Θ of order 1 producing the balance system. For this form $\check{L} = K$, $\Pi_i^\mu = F_i^\mu - K_{,z^i_\mu}$.*

Consider now a representation of density/flux tensor F_i^μ as the sum of vertical differential $d_v\check{L}$ and the complemental term

$$F_i^\mu = \check{L}_{,z^i_\mu} + \tilde{F}_i^\mu. \tag{5.3}$$

Then, a Lepage form obtained by using (4.1), (4.7) in (4.8) takes the form

$$\Theta = \left(\check{L}\eta + \check{L}_{,z^i_\mu}\omega^i \wedge \eta_\mu\right) + \left(\tilde{F}_i^\mu \omega^i \wedge \eta_\mu - \tilde{F}_i^\mu \omega^i_\mu \wedge \eta - \Pi_i \omega^i \wedge \eta\right) + 2\,\mathrm{Con}. \tag{5.4}$$

As a result we get the following

Proposition 2. *Any form of Lepage type of order* 1 *can be presented as the sum of Poincaré–Cartan form of Lagrangian problem and the complemental Poincaré–Cartan form of ML type.*

Corresponding balance system (4.10) *takes the form*

$$E_i(L)\left(j^k s(x)\right) + E_i(\tilde{F}, \Pi)\left(j^k s(x)\right) = 0, \tag{5.5}$$

where

$$\begin{cases} E_i(L)\left(s(x)\right) = d_\mu(L_{,z^i_\mu})\left(j^k s(x)\right) - L_{,y^i}\left(j^k s(x)\right), \\ E_i(\tilde{F}, \Pi) = d_\mu\left(\tilde{F}_i^\mu\right)\left(j^k s(x)\right) - \Pi_i\left(j^k s(x)\right). \end{cases} \tag{5.6}$$

Remark 2. This decomposition illustrates the naturality of Poincaré–Cartan forms of ML-type. Notice, though, that if one starts with the balance system in the form (4.10) containing F_i^μ and the source term K_i is NOT presented in the form $L_{,y^i} + \Pi_i$, decomposition (4.13) is not unique.

Remark 3. Remind that in 3-dim Euclidean space any covector field (1-form ω) is the sum of differential and codifferential forms: $\omega = dL + \delta K$ (Helmholtz decomposition). Corresponding balance system of the form (4.10) is the sum of conventional EL-system and the complemental one, with “antisymmetric source”. In other words

$$\Pi_i = -\check{L}_{,y^i} + (\delta\Omega)_i$$

for some two-form $\Omega = \Omega_{ij}\, dy^i \wedge dy^j$.

Consider now the general case of order k, i.e. assume that all the constitutive components of the balance system (3.1) are defined at the bundle $J^k(\pi)$. Restrict to the case where $F_i^\Lambda = 0$ for $|\Lambda| > 1$. Then the condition (4.7) reduces to

$$\begin{cases} \check{L}_{,z^i_\lambda} + \Pi_i^\lambda - F_i^\lambda = 0, & \text{for } |\Lambda| = 1, \\ \check{L}_{,z^i_\Lambda} + \Pi_i^\Lambda = 0, & \text{for } |\Lambda| > 1. \end{cases} \tag{5.7}$$

It follows from this (for arbitrary k) that

Theorem 4. *For any balance system written in the form* (4.10) *there exists* unique, mod 2 Con terms, *the form $\widetilde{\Theta}$ of Lepage type such that $F_i^\Lambda = 0$ for $|\Lambda| > 1$ producing this system. For this form conditions* (4.15) *are of the form*

$$\begin{cases} \Pi_i^\lambda = F_i^\lambda - \check{L}_{,z^i_\lambda}, \\ \Pi_i^\Lambda = -\check{L}_{,z^i_\Lambda}, \quad \textit{for } |\Lambda| > 1. \end{cases} \tag{5.8}$$

This form is

$$\widetilde{\Theta} = \check{L}\eta + \sum_{i,\mu} F_i^\mu \omega^i \wedge \eta_\mu - \Pi_i \omega^i \wedge \eta - F_i^\lambda \omega^i_\lambda \wedge \eta + \sum_{i,\Lambda,|\Lambda|\geqslant 0} \check{L}_{,z^i_\Lambda} \omega^i_\Lambda \wedge \eta. \tag{5.9}$$

Thus, in the general case we also have two special classes of forms Θ of Lepage type:

(1) ML (**modified lifted**)-forms $\widetilde{\Theta}$ satisfying to the conditions (4.16) and, such that $\check{L} = 0$.
(2) Lagrange form of the first order with $F_i^\mu = \check{L}_{,z^i_\mu}$, $\Pi_i = 0$.

6. First Noether theorem

Let

$$\Theta = \Theta^n + \Theta^{n+1} = \check{L}\eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F_i^{\Lambda\mu} \omega^i_\Lambda \wedge \eta_\mu - \sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi_i^\Lambda \omega^i_\Lambda \wedge \eta + 2\,\mathrm{Con} \tag{6.1}$$

be an $n + (n+1)$ form of Lepage type (4.8). Let a vector field $\xi \in \mathcal{X}(J^\infty(\pi))$ be an infinitesimal symmetry of the Θ in the sense that:

$$\mathcal{L}_\xi \Theta = d\alpha + \mathrm{Con} \quad \Leftrightarrow \quad \begin{cases} \mathcal{L}_\xi \Theta^n = d\alpha^{n-1} + \mathrm{Con}^n, \\ \mathcal{L}_\xi \Theta^{n+1} = d\alpha^n + \mathrm{Con}^{n+1}. \end{cases} \tag{6.2}$$

Here $\alpha = \alpha^{n-1} + \alpha^n$ is a form in $J^\infty(\pi)$.

Equality (6.2) for Θ^n can be presented in the form

$$d\alpha^{n-1} + \mathrm{Con}^n = (i_\xi d + d i_\xi)\Theta^n = i_\xi\big(d\Theta^n - \Theta^{n+1}\big) + d i_\xi \Theta^n + i_\xi \Theta^{n+1} = i_\xi \tilde{d}\Theta + d i_\xi \Theta^n + i_\xi \Theta^{n+1}.$$

Let now $s : U \to Y$ be a solution of the system of Eqs. (4.9). Taking in the last equation the pullback by the jet $j^\infty s$ of section s we get

$$\big(j^\infty s\big)^* d\alpha^{n-1} = \big(j^\infty s\big)^* d i_\xi \Theta^n + \big(j^\infty s\big)^* i_\xi \Theta^{n+1}, \tag{6.3}$$

or

$$d\big[\big(j^\infty s\big)^*\big(d i_\xi \Theta^n - \alpha^{n-1}\big)\big] = -\big(j^\infty s\big)^*\big(i_\xi \Theta^{n+1}\big). \tag{6.4}$$

This proves the first form of Noether equality in the following theorem.

Theorem 5. *Let $\xi \in \mathcal{X}(J^\infty(\pi))$ be an infinitesimal symmetry of the form $\Theta^n = \check{L}\eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F_i^{\Lambda\mu} \omega^i_\Lambda \wedge \eta_\mu$, then*

(1) *For any solution $s : U \to Y$ of the system (4.9),*

$$d\big[\big(j^\infty s\big)^*\big(i_\xi \Theta^n - \alpha^{n-1}\big)\big] = -\big(j^\infty s\big)^*\big(i_\xi \Theta^{n+1}\big). \tag{6.5}$$

(2) *In explicit form*

$$d\Bigg[\big(j^\infty s\big)^*\Bigg(i_\xi\Bigg[\check{L}\eta + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F_i^{\Lambda\mu} \omega^i_\Lambda \wedge \eta_\mu\Bigg] - \alpha^{n-1}\Bigg)\Bigg] = \big(j^\infty s\big)^*\Bigg(i_\xi\Bigg(\sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi_i^\Lambda \omega^i_\Lambda \wedge \eta\Bigg)\Bigg). \tag{6.6}$$

(3) *In terms of* Con*-differential, (6.5) has the form of "conservation law"*

$$j^\infty s^* \tilde{d}\big[\big(i_\xi \Theta^n - \alpha^{n-1}\big) - i_\xi \Theta^{n+1}\big] = 0. \tag{6.7}$$

(3) *Calculating interior derivative in (6.6) we write it in the form*

$$d\Bigg[\big(j^\infty s\big)^*\Bigg(\check{L}\xi^\mu \eta_\mu + \sum_{i,\mu,\Lambda,|\Lambda|\geqslant 0} F_i^{\Lambda\mu} \omega^i_\Lambda(\xi)\eta_\mu - \alpha_h^{n-1}\Bigg)\Bigg] = \big(j^\infty s\big)^*\Bigg(\sum_{i,\Lambda,|\Lambda|\geqslant 0} \Pi_i^\Lambda \omega^i_\Lambda(\xi)\eta\Bigg). \tag{6.8}$$

Here α_h^{n-1} is the horizontal component of α^{n-1}.

(4) *For an ML-form* (5.9) *Noether equality* (6.6) *has the form*

$$d\Big[(j^\infty s)^*\Big(i_\xi\Big(\check{L}\eta+\sum_{i,\mu}F_i^\mu\omega^i\wedge\eta_\mu\Big)-\alpha^{n-1}\Big)\Big]$$
$$=(j^\infty s)^*\Big(i_\xi\Big(\Pi_i\omega^i\wedge\eta+F_i^\lambda\omega^i_\lambda\wedge\eta-\sum_{i,\Lambda,|\Lambda|\geqslant 0}\check{L}_{,z^i_\Lambda}\omega^i_\Lambda\wedge\eta\Big)\Big). \tag{6.9}$$

Proof. Second representation of Noether equality follows from the first one and (6.1). □

Remark 4. Notice that although derivatives $L_{,z^i_\Lambda}, |\Lambda|$ do not enter Eq. (4.10) explicitly, they appear in the Noether balance laws.

Let now vector field ξ be an infinitesimal symmetry of the form Θ^{n+1}. Condition (6.2) for this form can be written in the form

$$(di_\xi+i_\xi d)\Theta^{n+1}=d\alpha^n+\mathrm{Con}^{n+1}\quad\Leftrightarrow\quad d\big(i_\xi\Theta^{n+1}-\alpha^n\big)+i_\xi\,d\Theta^{n+1}=\mathrm{Con}^{n+1}\,. \tag{6.10}$$

On the other side,

$$d\Theta^{n+1}=d\big(-\Pi_i^\Lambda\omega^i_\Lambda\wedge\eta\big)=-d\Pi_i^\Lambda\wedge\omega^i_\Lambda\wedge\eta-\Pi_i^\Lambda\,d\omega^i_\Lambda\wedge\eta.$$

Since $d\omega^i_\Lambda=-\omega^i_{\Lambda\mu}\wedge dx^\mu$, the last term in the right side vanishes. Decomposing $d\Pi_i^\Lambda=d_h\Pi_i^\Lambda+d_v\Pi_i^\Lambda$ we see that the form $d\Theta^{n+1}$ is 2-contact. Then, the form $i_\xi d\Theta^{n+1}$ is contact and from the equality (6.10) we conclude that

$$d\big(i_\xi\Theta^{n+1}-\alpha^n\big)=\mathrm{Con}\quad\Rightarrow\quad d_h\big(i_\xi\Theta^{n+1}-\alpha^n\big)=0. \tag{6.11}$$

As a result, the form $Q(\xi)=\pi^*_{21}i_\xi\Theta^{n+1}$ defines the class of horizontal cohomology (cohomology of the complex $(\bigwedge^*(J^\infty(\pi)),d_h)$) ($\mathfrak{g}$**-charge of the source** Θ^{n+1}) and if this class is zero, then, $Q(\xi)=d_h\Phi(\xi)\ \Leftrightarrow\ Q(\xi)=d\Phi(\xi)+\mathrm{Con}$. As a result, we got the proof of first two statements of the next theorem. Other statements follow trivially from the proved ones.

Theorem 6. *Let $\mathfrak{g}\subset\mathcal{X}(Y)$ is a Lie algebra of infinitesimal symmetries of the form Θ^{n+1}, see* (6.1)*. Let $\xi\in\mathfrak{g}$, then*

(1) $$di_\xi\Theta^{n+1}=\mathrm{Con}\quad\Rightarrow\quad d_h\big(i_\xi\Theta^{n+1}-\alpha^n\big)=0$$
and, therefore, for all sections $s\in\Gamma(\pi)$, $dj^\infty(s)^*i_\xi\Theta^{n+1}=0$.

(2) *n-form $Q(\xi)=i_\xi\Theta^{n+1}$ defines the class of cohomology $[Q(\xi)]\in H^n_h(J^\infty(\pi))$ in the horizontal complex $(\bigwedge^*(J^\infty(\pi)),d_h)$.*

(3) *If the class $[Q(\xi)]=0$, there exists a $\mathfrak{g}^*$-valued $(n+1)$-form $\Phi(\xi)$* ($\mathfrak{g}$-potential of the source Θ^{n+1})*, linearly depending on ξ, and such that $Q(\xi)=d\Phi(\xi)+\mathrm{Con}$. In this case locally (and in a topologically trivial domain, globally)*

$$j^\infty(s)^*i_\xi\Theta^{n+1}=d\big(j^\infty s\big)^*\Phi(\xi).$$

(4) *In the last case, for all solutions $s(x)$ of the system* (4.10) *and for all $\xi\in\mathfrak{g}$, the following conservation law holds*

$$d\big(j^\infty s\big)^*\big(i_\xi\Theta^n-\alpha^{n-1}-\Phi(\xi)\big)=0.$$

7. Conclusion

In this paper we defined a class of $(n+(n+1))$-exterior forms on the infinite jet bundle $J^\infty(\pi)$ of a fibred bundle $\pi:Y^{n+m}\to X^n$ satisfying the condition of Lepage type. Such a form Θ defines, in a conventional way of the infinitesimal variational calculus, the system of m partial differential equations having a form of balance laws. Thus, these *forms of Lepage type* can be used to present the balance system of continuum thermodynamics. The first Noether theorem (see [12]) was proved for such a balance system with an infinitesimal symmetry Lie algebra. We introduced a convenient class of forms of Lepage type — MP-forms and proved that any balance system can be presented as the Euler–Lagrange system corresponding to such a form. It was shown that any form of Lepage type can be presented as the sum of MP-form and the conventional Poincaré–Cartan form corresponding to a Lagrangian L.

In the continuation of this work we will study the presentation of a general systems as the sum of a conventional (corresponding to a Lagrangian) Poincaré–Cartan form and the complemental form of MP type. It would be interesting to see if such a splitting can be done in some canonical way and if the qualitative characteristics of the original system (symmetry groups, type of the system: hyperbolic, hyperbo-parabolic, etc.) are inherited (completely or partially) by the components of this decomposition?

It would also be interested to see the place of other canonical Lepage forms — Betunes–Krupka form and the Caratheodory form in this scheme.

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Appendix A. Con-differential

Differential $\tilde{d}$ known in topology under the name "con-differential" is defined on the couples of exterior forms of consecutive order.

Definition 2. For $k > 0$ define the operator

$$\tilde{d}\colon\ \Omega^{k+(k+1)} = \Omega^k(X) \oplus \Omega^{k+1}(X) \quad\to\quad \Omega^{(k+1)+(k+2)} = \Omega^{k+1}(X) \oplus \Omega^{k+2}(X)$$

as follows:

$$\tilde{d}\big(\alpha^k + \beta^{k+1}\big) = (-d\alpha + \beta) + d\beta. \tag{A.1}$$

Lemma 1. *$\tilde{d} \circ \tilde{d} = 0$.*

Proof. We have

$$\tilde{d}\tilde{d}\big(\alpha^k + \beta^{k+1}\big) = \tilde{d}\big((-d\alpha + \beta) + d\beta\big) = \big[-d(-d\alpha + \beta) + d\beta\big] + d(d\beta) = -d\beta + d\beta + 0. \qquad \square$$

Lemma 2 *(Poincaré). If $\tilde{d}(\alpha^k + \beta^{k+1}) = 0$, then, locally, $\alpha^k + \beta^{k+1} = \tilde{d}(\xi^{n-2} + \gamma^{k-1})$ for some $\xi^{k-2} \in \Omega^{k-2}$, $\gamma \in \Omega^{k-1}$.*

Proof.

$$\tilde{d}\big(\alpha^k + \beta^{k+1}\big) = 0 \quad\Leftrightarrow\quad \begin{cases} \beta - d\alpha = 0, \\ d\beta = 0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} \beta = d\alpha, \\ \beta^{k+1} = d\gamma^k \end{cases} \quad\Rightarrow\quad d\big(\alpha^k - \gamma^k\big) = 0 \quad\Rightarrow\quad \alpha^k - \gamma^k = d\xi^{k-1}$$
$$\Rightarrow\quad \alpha^k = d\xi^{k-1} + \gamma^k \quad\Rightarrow\quad \alpha^k + \beta^{k+1} = \big(-d\big(-\xi^{k-1}\big) + \gamma^k\big) + d\gamma^k = \tilde{d}\big(-\xi^{k-1} + \gamma^k\big). \qquad \square \tag{A.2}$$

Introduce the **shifted double of the Rham complex** $DR(X)$ of a manifold X complex of the couples of forms $\alpha^k + \beta^{k+1}$

$$0 \to \Omega^1(X) \oplus \Omega^0(X) \to \cdots \to \Omega^k(X) \oplus \Omega^{k-1}(X) \to \cdots \to \Omega^n(X) \oplus \Omega^{n-1}(X) \to 0 \oplus \Omega^n(X) \to . \tag{A.3}$$

This complex can be considered as **dual to the complex of chains generated by couples** (C^{k+1}, C^k) of submanifolds $C^{k+1} \subset X^n \ni C^k$ of dimensions $(k+1)$ and k: Duality is defined by integration

$$\big\langle \alpha^k + \beta^{k+1}, \big(C^{k+1}, C^k\big)\big\rangle = \int\limits_{C^{k+1}} \beta + \int\limits_{C^k} \alpha.$$

We have, the following properties whose proof is simple:

(1) For all submanifolds C^{k+2}.

$$\big\langle \tilde{d}\big(\alpha^k + \beta^{k+1}\big), \big(C^{k+2}, -\partial C^{k+2}\big)\big\rangle = 0.$$

Proof.

$$\begin{aligned}\big\langle \tilde{d}\big(\alpha^k + \beta^{k+1}\big), \big(C^{k+2}, -\partial C^{k+2}\big)\big\rangle &= \big\langle\big(\beta^{k+1} - d\alpha^k + d\beta^{k+1}\big), \big(C^{k+2}, -\partial C^{k+2}\big)\big\rangle \\ &= \int\limits_C d\beta^{k+1} - \int\limits_{\partial C} \big(\beta^{k+1} - d\alpha^k\big) = \int\limits_{\partial C} \beta^{k+1} - \int\limits_{\partial C} \beta^{k+1} + \int\limits_{\partial\partial C} \alpha^k = 0.\end{aligned} \tag{A.4}$$

(2) Equality $\langle \tilde{d}(\alpha^k + \beta^{k+1}), (C_1^{k+2}, C_2^{k+1})\rangle = 0$ for all couples of submanifolds C_1^{k+2}, C_2^{k+1} is valid if and only if $\beta = d\alpha$.

Proof. Let

$$0 = \big\langle \tilde{d}\big(\alpha^k + \beta^{k+1}\big), \big(C_1^{k+2}, C_2^{k+1}\big)\big\rangle = \int\limits_{C_1} d\beta + \int\limits_{C_2} (\beta - d\alpha)$$

for all C_1, C_2. Taking $C_1 = \emptyset$ we get $\int_{C_2} (\beta - d\alpha)$ for all submanifolds C_2. This is possible if and only if $\beta - d\alpha = 0$. Vice versa, if $\beta = d\alpha$, then $d\beta = 0$ and $\tilde{d}(\alpha^k + \beta^{k+1}) = (\beta - d\alpha) + d\beta = 0$.

(3) Let $\langle \tilde{d}(\alpha^k + \beta^{k+1}), (C_1^{k+2}, -C_2^{k+1})\rangle = 0$ for all $\alpha^k + \beta^{k+1}$, then $C_2 = \partial C_1$.

Proof.

$$\left\langle \tilde{d}\left(\alpha^k + \beta^{k+1}\right), \left(C_1^{k+2}, -C_2^{k+1}\right)\right\rangle = \int_{C_1} d\beta + \int_{C_2} (\beta - d\alpha) = \int_{C_2 - \partial C_1} \beta + \int_{\partial C_2} \alpha.$$

If the last expression is equal zero for all C_1, C_2, then taking $\alpha = 0$ we get $C_2 = \partial C_1$.

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